

# Minimal Length and the Quantum Bouncer: A Nonperturbative Study

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## Abstract

We present the energy eigenvalues of a quantum bouncer in the framework of the Generalized (Gravitational) Uncertainty Principle (GUP) via quantum mechanical and semiclassical schemes. In this paper, we use two equivalent nonperturbative representations of a deformed commutation relation in the form  $[X, P] = i\hbar(1 + \beta P^2)$  where  $\beta$  is the GUP parameter. The new representation is formally self-adjoint and preserves the ordinary nature of the position operator. We show that both representations result in the same modified semiclassical energy spectrum and agrees well with the quantum mechanical description.

*Keywords:* Quantum Gravity; Generalized Uncertainty Principle; Quantum Bouncer; Minimal Length.

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## 1 Introduction

Various proposals for quantum gravity such as string theory, loop quantum gravity, quantum geometry, and black-hole physics all agree on the the existence of a minimum measurable length of the order of the Planck length  $\ell_{Pl} = \sqrt{G\hbar/c^3} \approx 10^{-35}m$  where  $G$  is Newton's gravitational constant. It can solve the renormalizability problem of relativity which is related to the field theoretical ultraviolet divergences in the theory. Indeed, beyond the Planck energy scale, the effects of gravity are so important which would result in discreteness of the very spacetime. However, by introducing a minimal length as an effective cutoff in the ultraviolet domain, the theory can be renormalizable.

The introduction of this idea in the form of the Generalized Uncertainty Principle (GUP) has attracted much attention in recent years and many papers have been appeared in the literature to address the effects of GUP on various quantum mechanical systems [1–17]. In doubly special relativity theories,

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we consider the Planck energy as an additional invariant beside the velocity of light [18–22]. A GUP which is consistent with these theories is also discussed in Refs. [23–27].

In this paper, we consider the problem of the quantum bouncer in the context of the generalized uncertainty principle in the form  $[X, P] = i\hbar(1 + \beta P^2)$  where  $\beta$  is the GUP parameter. In ordinary quantum mechanics, the energy levels of a test particle which is bouncing elastically and vertically on the Earth’s surface should be quantized as a manifestation of the particle-wave duality. This effect has been confirmed a few years ago using a high precision neutron gravitational spectrometer. Using the perturbative representations, the effects of the minimal length and/or maximal momentum on the quantum bouncer spectrum are also discussed in Refs. [27–29]. Here, we use two equivalent nonperturbative representations which exactly satisfy the modified commutation relation and find the GUP corrected energy spectrum of the quantum bouncer. Therefore, we can relax the assumption of the smallness of the GUP parameter and obtain the solutions for arbitrary values of  $\beta$ . We show that the semiclassical results agree well with the quantum mechanical results even for the low lying states.

## 2 The Generalized Uncertainty Principle

Let us consider a generalized uncertainty principle which results in a minimum observable length

$$\Delta X \Delta P \geq \frac{\hbar}{2} (1 + \beta(\Delta P)^2 + \zeta), \quad (1)$$

where  $\beta$  is the GUP parameter and  $\zeta$  is a positive constant that depends on the expectation values of the momentum operator. We also have  $\beta = \beta_0/(M_{Pl}c)^2$  where  $M_{Pl}$  is the Planck mass and  $\beta_0$  is of order one. It is straightforward to check that the inequality relation (1) implies the existence of a minimum observable length as  $(\Delta X)_{min} = \hbar\sqrt{\beta}$ . In the context of string theory, we can interpret this length as the length of the string where it is proportional to the square root of the GUP parameter. In one dimension, the above uncertainty relation can be obtained from the following deformed commutation relation

$$[X, P] = i\hbar(1 + \beta P^2), \quad (2)$$

where for  $\beta = 0$  we recover the well-known commutation relation in ordinary quantum mechanics. Now using Eqs. (1) and (2), we find the explicit form of  $\zeta$  as the expectation value of the momentum operator i.e.  $\zeta = \beta \langle P \rangle^2$ . As Kempf, Mangano, and Mann (KMM) have suggested in their seminal paper, in momentum space representation, we can write  $X$  and  $P$  as [30]

$$P = p, \quad (3)$$

$$X = i\hbar (1 + \beta p^2) \frac{\partial}{\partial p}, \quad (4)$$

where  $X$  and  $P$  are symmetric operators on the dense domain  $S_\infty$  with respect to the following scalar product:

$$\langle \psi | \phi \rangle = \int_{-\infty}^{+\infty} \frac{dp}{1 + \beta p^2} \psi^*(p) \phi(p). \quad (5)$$

Also we have  $\int_{-\infty}^{+\infty} \frac{dp}{1 + \beta p^2} |p\rangle \langle p| = 1$  and  $\langle p | p' \rangle = (1 + \beta p^2) \delta(p - p')$ . With this definition, the commutation relation (2) is exactly satisfied. Note that the KMM representation is not unique and we can use another representation using appropriate canonical transformations. For instance, consider the following exact representation

$$X = x, \quad (6)$$

$$P = \frac{\tan(\sqrt{\beta} p)}{\sqrt{\beta}}, \quad (7)$$

where  $x$  and  $p$  obey the canonical commutation relation  $[x, p] = i\hbar$ . Note that, in contrary to the KMM representation, this representation is formally self-adjoint and preserves the ordinary nature of the position operator. Moreover, this definition exactly satisfies the condition  $[X, P] = i\hbar(1 + \beta P^2)$  and agrees with the well-known perturbative proposal [29, 31, 32]

$$X = x, \quad (8)$$

$$P = p \left( 1 + \frac{1}{3} \beta p^2 \right), \quad (9)$$

to the first order of the GUP parameter. Indeed, the two representations are equivalent and they are related by the following canonical transformation:

$$X \rightarrow \left[ 1 + \arctan^2 \left( \sqrt{\beta} P \right) \right] X, \quad (10)$$

$$P \rightarrow \arctan \left( \sqrt{\beta} P \right) / \sqrt{\beta}, \quad (11)$$

which transforms Eqs. (6) and (7) to Eqs. (3) and (4) subjected to Eq. (2). We can interpret  $P$  and  $p$  as follows:  $p$  is the momentum operator at low energies ( $p = -i\hbar\partial/\partial x$ ) and  $P$  is the momentum operator at high energies. Obviously, this procedure affects all Hamiltonians in quantum mechanics. Consider the following Hamiltonian:

$$H = \frac{P^2}{2m} + V(X), \quad (12)$$

which using Eqs. (6) and (7) can be written exactly and perturbatively as

$$H = \frac{\tan^2(\sqrt{\beta}p)}{2\beta m} + V(x), \quad (13)$$

$$= H_0 + \sum_{n=3}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n} - 1) (2n - 1) B_{2n}}{2m(2n)!} \beta^{n-2} p^{2(n-1)}, \quad (14)$$

where  $H_0 = p^2/2m + V(x)$  and  $B_n$  is the  $n$ th Bernoulli number. The corrected terms in the modified Hamiltonian are only momentum dependent and are proportional to  $p^{2(n-1)}$  for  $n \geq 3$ . In fact, the presence of these terms leads to a positive shift in the energy spectrum. In the quantum domain, this Hamiltonian results in the following generalized Schrödinger equation in quasiposition representation

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + \sum_{n=3}^{\infty} \alpha_n \hbar^{2(n-1)} \beta^{n-2} \frac{\partial^{2(n-1)} \psi(x)}{\partial x^{2(n-1)}} + V(x) \psi(x) = E \psi(x), \quad (15)$$

where  $\alpha_n = 2^{2n} (2^{2n} - 1) (2n - 1) B_{2n} / 2m(2n)!$  and the second term is due to the GUP corrected terms in Eq. (14).

### 3 The Quantum Bouncer

Although the quantum stationary states of matter in the gravitational field is theoretically possible, the observation of such quantum effect is very difficult experimentally. This is due to the fact that

the gravitational quantum effects are negligible for macroscopic objects and for charged particles the electromagnetic interaction has the dominate role. So we should use neutral elementary particles with a long lifetime such as neutrons. A few years ago, a famous measurement is performed by Nesvizhevsky et al. using a high precision neutron gravitational spectrometer [33–35]. In this experiment, the quantization of energy of ultra cold neutrons bouncing above a mirror in the Earth’s gravitational field was demonstrated.

### 3.1 Exact solutions

Here, we are interested to study the effects of the minimal length on the energy spectrum of a particle of mass  $m$  which is bouncing vertically and elastically on a reflecting hard floor so that

$$V(X) = \begin{cases} mgX & \text{for } X > 0, \\ \infty & \text{for } X \leq 0, \end{cases} \quad (16)$$

where  $g$  is the acceleration in the Earth’s gravitational field. In ordinary quantum mechanics ( $\beta = 0$ ), this problem is exactly solvable and the eigenfunctions can be written in the form of the Airy functions. Moreover, the energy eigenvalues are related to the zeros of the Airy function. In the semiclassical approximation, the energy eigenvalues can be obtained using the Bohr-Sommerfeld formula as

$$E_n \simeq \left( \frac{9m}{8} \left[ \pi \hbar g \left( n - \frac{1}{4} \right) \right]^2 \right)^{1/3}, \quad (17)$$

which is valid with a high degree of accuracy of  $\sim 1\%$  even for the lowest quantum state. However, for  $\beta \neq 0$  the situation is quite different.

In the presence of the minimal length, it is more appropriate to study the problem in the momentum space. Because of the linear form of the potential, the generalized Schrödinger equation can be cast into a first-order differential equation in this space. Now we define a new variable  $z = x - \frac{E}{mg}$  and rewrite the generalized Schrödinger equation (13) in the momentum space as

$$\tan^2(\sqrt{\beta}p) \phi(p) + 2i\beta m^2 g \hbar \phi'(p) = 0, \quad (18)$$

where the prime denotes the derivative with respect to  $p$ . This equation admits the following solution

$$\phi(p) = \phi_0 \exp \left[ \frac{i\alpha}{\beta \hbar} \left( \frac{\tan(\sqrt{\beta}p)}{\sqrt{\beta}} - p \right) \right]. \quad (19)$$

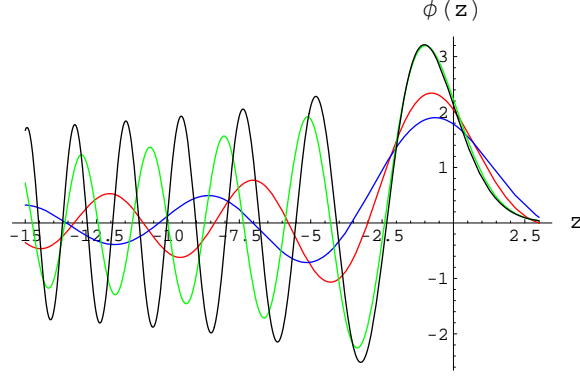


Figure 1: The wave function  $\phi(z)$  for  $\beta = 0$  (black line),  $\beta = 0.1$  (green line),  $\beta = 1$  (red line), and  $\beta = 2$  (blue line). The eigenenergies are the minus of the zeros of  $\phi(z)$  and  $\alpha = \hbar = 1$ .

where  $\phi_0$  is a constant and  $\alpha^{-1} = 2m^2g$ . To proceed further and for the sake of simplicity, let us work in the units of  $\alpha = \hbar = 1$ . Now the energy eigenvalues are the minus of the roots of the following wave function in the quasiposition space representation

$$\phi(z) = \phi_0 \int_{-\frac{\pi}{2\sqrt{\beta}}}^{\frac{\pi}{2\sqrt{\beta}}} \exp \left[ \frac{i}{\beta} \left( \frac{\tan(\sqrt{\beta}p)}{\sqrt{\beta}} - p \right) \right] e^{ipz} dp. \quad (20)$$

To evaluate the integral, it is more appropriate to use the new variable  $u = \tan(\sqrt{\beta}p)/\sqrt{\beta}$  and write the wave function as

$$\phi(z) = \phi_0 \int_{-\infty}^{\infty} \frac{(1 + i\sqrt{\beta}u)^{\frac{z-1/\beta}{\sqrt{\beta}}}}{(1 + \beta u^2)^{1 + \frac{z-1/\beta}{2\sqrt{\beta}}}} e^{i\frac{u}{\beta}} du. \quad (21)$$

This integral can be evaluated numerically and the resulting solutions for  $\beta = \{0, \frac{1}{10}, 1, 2\}$  are depicted in Fig. 1. As the figure shows, the energy spectrum increases in the presence of the minimal length i.e.  $\beta \neq 0$ . The calculated energy eigenvalues for the first ten states are also shown in Table 1 which confirms the positive shift in the energy levels of the quantum bouncer in the GUP framework.

To explain how the energy spectrum of the quantum bouncer increases in the GUP scenario, let us study the dynamics of a particle in the classical domain as

$$\frac{dP}{dt} = \{P, H\}, \quad (22)$$

where the Poisson bracket in classical mechanics corresponds to the quantum mechanical commutator

$n$	$\beta = 0$	$\beta = 1$
1	2.33811	3.00000
2	4.08795	5.77871
3	5.52056	8.36567
4	6.78671	10.8569
5	7.94413	13.2878
6	9.02265	15.6765
7	10.0402	18.0333
8	11.0085	20.3649
9	11.9360	22.6761
10	12.8288	24.9702

Table 1: The first ten quantized energies of a bouncing particle in GUP formalism for  $\alpha = \hbar = 1$ .

via

$$\frac{1}{i\hbar}[\hat{A}, \hat{B}] \Rightarrow \{A, B\}. \quad (23)$$

Now for the quantum bouncer we have

$$\frac{dP}{dt} = mg\{P, X\} = -mg(1 + \beta P^2) = -mG(P), \quad (24)$$

where  $G(P) = g(1 + \beta P^2) > g$  is the effective gravitational acceleration. So an extra  $P^2$  dependent force  $F' = -\beta mgP^2$  acts on the particle which increases its energy in the presence of the minimal length.

It is worth to mention that the results of the Nesvizhevsky's experiment have a quite large error bars. If we consider  $\beta$  as a universal constant, for example the value that Brau has found by studying the hydrogen atom, we obtain [6, 28]

$$\beta < 2 \times 10^{-5} \text{ fm}^2, \quad \text{and} \quad \Delta E_1 \simeq \Delta E_2 < 10^{-19} \text{ peV}. \quad (25)$$

These upper bounds tell us that the effects of the minimal observable length are largely unobservable in the experiment, since the maximal precision is  $10^{-2}$  peV [35].

In analogy with electrodynamics, the observation of spontaneous decay of an excited state in Nesvizhevsky's experiment could demonstrates quantum behavior of the gravitational field as a manifestation of the Planck-scale effect [36]. Since the energy spectrum of the quantum bouncer increases in the presence of the minimal length, we expect that the rate of this decay changes as a trace of quantum gravitational effects via existence of a minimal length scale. The transition probability in the quadrupole approximation

and in the presence of GUP is [27, 29]

$$\Gamma_{k \rightarrow n}^{(\text{GUP})} = \frac{512}{5} \frac{\left(\lambda_k^{(\text{GUP})} - \lambda_n^{(\text{GUP})}\right)^5}{(\lambda_k - \lambda_n)^8} \left(\frac{m}{M_{Pl}}\right)^2 \frac{E_0^5 c}{\gamma^4 (\hbar c)^5}, \quad (26)$$

where  $\gamma = (\alpha \hbar^2)^{-1/3}$ ,  $E_0 = mg/\gamma$ ,  $\lambda_n^{(\text{GUP})} = E_n/E_0$ , and  $-\lambda_n$  are the zeros of the Airy function. Although this rate is expected to be low, we hope to find the effects of the generalized uncertainty principle on the transition rate of ultra cold neutrons in the future more accurate experiments.

### 3.2 Semiclassical approximation

We can also estimate the energy spectrum using the semiclassical scheme. There are two options for writing the Hamiltonian of the quantum bouncer  $H = P^2/2m + mgX$  in the classical domain. The first one is based on the KMM's proposal which results in

$$\frac{p^2}{2m} + mg(1 + \beta p^2)x = E, \quad (27)$$

and the second one is based on Eqs. (6) and (7), i.e.,

$$\frac{\tan^2(\sqrt{\beta}p)}{2\beta m} + mgx = E. \quad (28)$$

Since both formulations are equivalent representations of the same algebra, we expect that they result in the same energy spectrum. The approximate energy eigenvalues can be obtained using Bohr-Sommerfeld quantization rule

$$\oint p \, dx = \left(n - \frac{1}{4}\right) h, \quad n = 1, 2, \dots, \quad (29)$$

where the presence of the WKB corrected term  $-1/4$  is due to the fact that the wave function can penetrate into the right hand classically forbidden region, but it is exactly zero for  $x \leq 0$ . Bohr-Sommerfeld formula for the first proposal (27) results in

$$\oint p \, dx = 2 \int_0^\epsilon \sqrt{\frac{\epsilon - x}{\alpha + \beta x}} \, dx, \quad (30)$$

where  $\epsilon = E^{(SC)}/mg$  is the rescaled semiclassical energy. For the second representation (28) we have

$$\oint p \, dx = \frac{2}{\sqrt{\beta}} \int_0^\epsilon \arctan\left(\sqrt{\frac{\beta}{\alpha}}(\epsilon - x)\right) \, dx. \quad (31)$$



It is straightforward to check that both integrals are equivalent and the energy eigenvalues are given by the roots of the following equation

$$(\alpha + \beta\epsilon) \arctan\left(\sqrt{\frac{\beta\epsilon}{\alpha}}\right) - \sqrt{\alpha\beta\epsilon} - \left(n - \frac{1}{4}\right) \pi\hbar\beta^{3/2} = 0. \quad (32)$$

To zeroth-order of the GUP parameter, this equation admits the solution which is given by Eq. (17), i.e.  $\epsilon_n^0 = \left((9/4)\alpha [\pi\hbar(n - 1/4)]^2\right)^{1/3}$ . Moreover, to the first-order we have the following solution:

$$\epsilon_n^1 = \epsilon_n^0 \left(1 + \frac{2}{15} \frac{\beta}{\alpha} \epsilon_n^0\right). \quad (33)$$

Figure 2 shows the schematic solutions of Eq. (32) for  $\beta = 1$  and  $\alpha = 1$ . In Table 2 we have reported the exact and semiclassical energy spectrum of the quantum bouncer in the GUP framework for  $\beta = 1$ . As the results show, the semiclassical solutions agree well with the exact ones up to a high degree of accuracy of  $\sim 0.1\% - 1\%$  even for the low energy quantum states.

### 3.3 WKB approximation

To check the validity of the Bohr-Sommerfeld quantization rule for this modified quantum mechanics, let us write the first-order generalized Schrödinger equation as

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + \frac{\hbar^4}{3m} \frac{\partial^4 \psi(x)}{\partial x^4} + V(x) \psi(x) = E \psi(x), \quad (34)$$

and take

$$\psi(x) = e^{i\Phi(x)}, \quad (35)$$

where  $\Phi(x)$  can be expanded as a power series in  $\hbar$  in the semiclassical approximation, i.e.,

$$\Phi(x) = \frac{1}{\hbar} \sum_{n=0}^{\infty} \hbar^n \Phi_n(x). \quad (36)$$

So we have

$$\frac{\partial^2 \psi(x)}{\partial x^2} = -(\Phi'^2 - i\Phi'') \psi(x), \quad (37)$$

$$\frac{\partial^4 \psi(x)}{\partial x^4} = (\Phi'^4 - 6i\Phi'^2\Phi'' - 3\Phi''^2 - 4\Phi''' \Phi' + i\Phi'''' ) \psi(x), \quad (38)$$

where  $\Phi'$  indicates the derivative of  $\Phi$  with respect to  $x$ . To zeroth-order ( $\Phi(x) \simeq \Phi_0(x)/\hbar$ ) and for  $\hbar \rightarrow 0$  we obtain

$$\Phi_0'^2 + \frac{2}{3}\beta\Phi_0'^4 = 2m(E - V(x)). \quad (39)$$

Now the comparison with Eq. (41) shows  $\Phi_0' = p$  and consequently

$$\psi(x) \simeq \exp \left[ \frac{i}{\hbar} \int p dx \right], \quad (40)$$

which is the usual zeroth-order WKB wave function obeying the Bohr-Sommerfeld quantization rule. The generalization of this result to higher orders perturbed Hamiltonians and the nonperturbative Hamiltonian (13) is also straightforward. Indeed, the agreement between the exact and semiclassical results is the manifestation of the validity of the Bohr-Sommerfeld quantization rule in this modified quantum mechanics.

### 3.4 Perturbative study

Now, following Ref. [28], let us study this problem perturbatively. To first-order of the GUP parameter the Hamiltonian can be written as

$$H = \frac{p^2}{2m} + \beta \frac{p^4}{3m} + V(x) = H_0 + \beta \frac{p^4}{3m}. \quad (41)$$

Using the first order time-independent perturbation theory, the energy spectrum is given by

$$E_n^1 \simeq E_n^0 + \Delta E_n, \quad (42)$$

where to the first order we have

$$\begin{aligned} \Delta E_n &= \frac{\beta}{3m} \langle p^4 \rangle = \frac{4}{3} \beta m \langle (H_0 - V)^2 \rangle, \\ &= \frac{4}{3} \beta m \left[ (E_n^0)^2 - 2E_n^0 \langle V(x) \rangle + \langle V(x)^2 \rangle \right]. \end{aligned} \quad (43)$$

Since  $V(x) \sim x$  is a power-law potential, the virial theorem gives

$$\langle V(x) \rangle = \frac{2}{3} E_n^0. \quad (44)$$

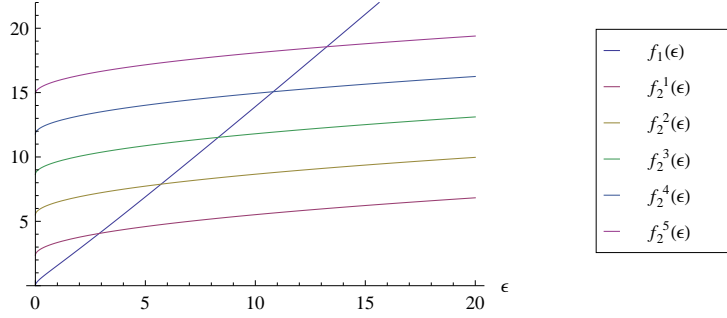


Figure 2: The schematic solutions of Eq. (32) for  $\beta = 1$  and  $\alpha = \hbar = 1$ .  $f_1(\epsilon) = (\alpha + \beta\epsilon) \arctan \sqrt{\beta\epsilon/\alpha}$  and  $f_2^n(\epsilon) = \sqrt{\alpha\beta\epsilon} + (n - 1/4) \pi \hbar \beta^{3/2}$ .

So using  $\langle V(x)^2 \rangle = \frac{8}{15}(E_n^0)^2$  [28], the relation (43) reduces to

$$\Delta E_n = \frac{4}{15} \beta (E_n^0)^2. \quad (45)$$

If we define  $\mathcal{E}_n = E_n/mg$  we obtain

$$\mathcal{E}_n^1 \simeq \mathcal{E}_n^0 \left( 1 + \frac{2}{15} \frac{\beta}{\alpha} \mathcal{E}_n^0 \right), \quad (46)$$

which has the form of the semiclassical solution (33) and  $\mathcal{E}_n^0$  is given by [28]

$$\mathcal{E}_n^0 = \alpha^{1/3} \lambda_n, \quad (47)$$

As stated before,  $\mathcal{E}_n^0$  are nearly equal to  $\epsilon_n^0$  even for the lowest quantum state. Therefore, our results agree with those of Ref. [28] up to the first order of the GUP parameter.

## 4 Conclusions

In this paper, we have investigated the effects of the generalized uncertainty principle on the energy spectrum of a quantum bouncer. Using two exact and equivalent nonperturbative representations, we found the generalized Schrödinger equation in the momentum space and obtained the corresponding energy eigenvalues and the eigenfunctions. The second representation, in contrary to the KMM representation, was formally self-adjoint and preserved the ordinary nature of the position operator. Then, using the proper semiclassical approximation, we found the almost accurate energy spectrum even for

$n$	$\epsilon_n^{\text{exact}}$	$\epsilon_n^{\text{SC}}$	$ \frac{\Delta\epsilon_n}{\epsilon_n} $
1	3.00000	2.90366	0.032
2	5.77871	5.71575	0.011
3	8.36567	8.31537	0.006
4	10.8569	10.8137	0.004
5	13.2878	13.2495	0.003
6	15.6765	15.6416	0.002
7	18.0333	18.0010	0.002
8	20.3649	20.3348	0.001
9	22.6761	22.6477	0.001
10	24.9702	24.9433	0.001

Table 2: The exact and semiclassical energy levels of the quantum bouncer for  $\beta = 1$  and  $\alpha = \hbar = 1$ .

the low lying eigenstates and similar to the perturbative representations we observed a positive shift in the energy levels. Also, we explicitly found the first order quantum mechanical energy spectrum and showed the validity of the Bohr-Sommerfeld quantization rule in this modified quantum mechanics.

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